JOURNAL OF COMPUTATIONAL PHYSICS 59, 490–492 (1985)

Note

A Modification of the Delves–Lyness Method for Locating the Zeros of Analytic Functions

A new formula is proposed for the calculation of the integrals involved in the application of the Delves–Lyness method for locating the zeros of analytic functions inside an arbitrary simple smooth contour. This formula does not contain the derivative of the analytic function whose zeros are sought and, moreover, is free from multivaluedness problems for the integrand. © 1985 Academic Press, Inc.

1. Introduction

We reconsider the Delves–Lyness method for locating the zeros of an analytic function $f(z)$ inside an arbitrary simple smooth contour C [3], assuming that $f(z)$ does not have zeros on C. (This method is also reported in [5] and, in the particular case of one zero, in [8].) A revised form of this method was recently proposed by Li [6]. Both in the original [3] and in the revised [6] form of the method the main task is to evaluate the integrals

$$s_k = \frac{1}{2\pi i} \int_C z^k \frac{f'(z)}{f(z)}\, dz, \qquad k = 0(1)\, n, \tag{1}$$

where $s_0 = n$ is the number of zeros of $f(z)$ inside C [3, 4]. Generally, s_k cannot be evaluated in closed form and the trapezoidal rule [2, 7] is used for their approximation [3].

The main difficulty in using (1) is that not only the analytic function $f(z)$ but its derivative $f'(z)$, too, should be evaluated. In order to avoid evaluating $f'(z)$, Delves and Lyness [3] proposed to take into account that

$$\frac{d}{dz} \log f(z) = \frac{f'(z)}{f(z)} \tag{2}$$

and perform an integration by parts in (1). The resulting algorithm is described in detail in [3], but it is somewhat complicated due partly to multivaluedness problems resulting from the nonanalyticity of $\log f(z)$. Recently, Carpentier and Dos Santos [1] proposed a second algorithm for evaluating s_k without using the derivative of $f(z)$ (or approximating it in some way). This algorithm is also

0021-9991/85 $3.00

somewhat complicated and, moreover, it was suggested for circular regions only. (We can add that some references where the Delves–Lyness method was applied to the solution of real physical and engineering problems are reported in [1].)

Here we propose a new formula for the integrals s_k ($k = 1, 2, \ldots, n$), based also on an integration by parts (as has been the case in [3]), but free from multivaluedness problems and restrictions on the contour C (as has been the case in [1]). We will not be concerned with the computation of the index n of $f(z)$ with respect to the contour C [4] (equal to the number of zeros of $f(z)$ inside C [4]), since this is an integer and it does not require an accurate computation. Generally, n is computed by using the formula [4]

$$n = \operatorname{Ind} f(z) = \frac{1}{2\pi}[\arg f(z)]_C. \tag{3}$$

Further methods for computing n are well known [1, 4].

2. The Proposed Formula

We will prove that

THEOREM 1. *The integrals s_k, defined by* (1), *are also given by*

$$s_k = -\frac{k}{2\pi i}\int_C z^{k-1}\log[(z-a)^{-n}f(z)]\,dz + na^k, \tag{4}$$

where a is an arbitrary point inside C.

Proof. At first, we notice that since the index of $(z-a)^{-n}$ is equal to $-n$ and the index of $f(z)$ is equal to n (with respect to the contour C), then the index of $(z-a)^{-n}f(z)$ is equal to 0 [4]. Therefore, $\log[(z-a)^{-n}f(z)]$ is a singlevalued function along the contour C. Now, we take into account that

$$\frac{d}{dz}\log[(z-a)^{-n}f(z)] = \frac{f'(z)}{f(z)} - \frac{n}{z-a} \tag{5}$$

and we perform an integration by parts in (4). Thus we have to prove that

$$s_k = \frac{1}{2\pi i}\int_C z^k\left[\frac{f'(z)}{f(z)} - \frac{n}{z-a}\right]dz + na^k. \tag{6}$$

But since

$$a^k = \frac{1}{2\pi i}\int_C \frac{z^k}{z-a}\,dz, \tag{7}$$

because of the Cauchy integral formula in complex analysis, (6) reduces to (1). This completes the proof of the theorem.

Remark 1. For $k=0$ (4) reduces to $s_0=n$ as was expected.

Remark 2. In order to evaluate $\log g(z)$, with $g(z)=(z-a)^{-n}f(z)$, in the computer in order to use it in (4), we note that $\log g(z)=\log|g(z)|+i\vartheta$, where $\vartheta=\tan^{-1}[\operatorname{Im} g(z)/\operatorname{Re} g(z)]+j\pi$, where j is usually a small integer which may vary along C. Although the selection of j at the first point of C, where $\log g(z)$ is evaluated, is essentially arbitrary, since

$$\int_C z^{k-1}\,dz=0, \qquad k=1,2,..., \tag{8}$$

yet, as $\log g(z)$ is computed further along C, attention should be paid so that ϑ varies continuously along the whole contour C. Clearly, in this case ϑ is a singlevalued function of the points of the contour C (as already explained); that is, its increment in traversing the whole contour C is equal to zero. This continuous variation of ϑ can be easily achieved in the computer (the fact that $f(z)$ was assumed without zeros along C taken always into consideration).

References

1. M. P. Carpentier and A. F. Dos Santos, *J. Comput. Phys.* **45** (1982), 210.
2. P. J. Davis and P. Rabinowitz, "Numerical Integration," pp. 53, 69, Blaisdell, Waltham, Mass., 1967.
3. L. M. Delves and J. N. Lyness, *Math. Comp.* **21** (1967), 543.
4. F. D. Gakhov, "Boundary Value Problems," pp. 85, 138, Pergamon, New York, 1966.
5. A. S. Householder, "The Numerical Treatment of a Single Nonlinear Equation," p. 191, McGraw-Hill, New York, 1970.
6. T.-Y. Li, *SIAM J. Numer. Anal.* **20** (1983), 865.
7. J. N. Lyness and L. M. Delves, *Math. Comp.* **21** (1967), 561.
8. J. E. McCune, *Phys. Fluids* **9** (1966), 2082.

Received March 7, 1984; revised August 21, 1984

N. I. Ioakimidis

Division of Applied Mathematics and Mechanics,
School of Engineering, University of Patras,
P.O. Box 1120, GR-261.10 Patras, Greece

and

E. G. Anastasselou

Division of Applied Mechanics,
The National Technical University of Athens,
P.O. Box 61028, GR-151.10 Amaroussion, Greece